

TOPOLOGICALLY SLICE KNOTS THAT ARE NOT SMOOTHLY SLICE IN ANY DEFINITE 4-MANIFOLD

KOUKI SATO

ABSTRACT. We prove that there exist infinitely many topologically slice knots which cannot bound a smooth null-homologous disk in any definite 4-manifold. Furthermore, we show that we can take such knots so that they are linearly independent in the the knot concordance group.

1. INTRODUCTION

A knot K in S^3 is called *smoothly slice* (*topologically slice*) if K bounds a smooth disk (resp. topologically locally flat disk) in B^4 . While any smoothly slice knot is obviously topologically slice, it has been known that there exist infinitely many topologically slice knots that are not smoothly slice (for instance, see [3, 6]). The purpose of this paper is to prove that there exist infinitely many topologically slice knots which cannot bound a null-homologous smooth disk not only in B^4 but also in any 4-manifold with definite intersection form.

For a 4-manifold V with boundary S^3 , we call a knot K in S^3 *smoothly slice in V* if K bounds a smooth disk D in V such that $[D, \partial D] = 0 \in H_2(V, \partial V; \mathbb{Z})$. We call a 4-manifold V *definite* if the intersection form of V is either positive definite or negative definite. We denote the smooth knot concordance group by \mathcal{C} . Then our main theorem is stated as follows.

Theorem 1. *There exist infinitely many topologically slice knots which are not slice in any definite 4-manifold. Furthermore, we can take such knots so that they are linearly independent in \mathcal{C} .*

In order to prove Theorem 1, we use the Heegaard Floer τ -invariant and the V_k -invariants defined in [14]. In particular, by combining Wu's cabling formula [22] and Bodnár-Celoria-Golla's connected sum formula [1] for V_k -invariants, we prove the following proposition. Here we denote the mirror image of a knot K by K^* , the $(n, 1)$ -cable of K by $K_{n,1}$ and the connected sum of two knots K and J by $K \# J$.

Proposition 1. *Let K and J be knots. If $V_0(K) > V_0(J)$ and $\tau(K), \tau(J) > 0$, then for any positive integer n with $\tau(K) < n \cdot \tau(J)$, the knot $K \# (J_{n,1})^*$ is not slice in any definite 4-manifold.*

Note that if both K and J are topologically slice, then $K \# (J_{n,1})^*$ is also topologically slice for any $n \in \mathbb{Z} \setminus \{0\}$. Furthermore, it follows from [10, Proposition 6.1] and [10, Theorem B.1] that for any $m \in \mathbb{Z}_{>0}$, there exists a topologically slice knot K_m with $V_0(K_m) = m$. Hence by taking $K_l \# ((K_m)_{n,1})^*$ so that $l > m$ and n is sufficiently large, we immediately obtain infinitely many topologically slice knots which are not slice in any definite 4-manifold. Our proof of the linear independence of these topologically slice knots relies on Kim-Park's recent result [12].

The problem of smooth sliceness leads to the notion of the *kinkiness* of knots, as defined by Gompf [6]. Let K be a knot in $S^3 = \partial B^4$, and consider all self-transverse immersed disks in B^4 with boundary K . Then we define $k_+(K)$ (resp. $k_-(K)$) to be the minimal number of positive (resp. negative) self-intersection points occurring in such a disk. Gompf proved in [6] that for any $n \in \mathbb{Z}_{>0}$, there exists a topologically slice knot K such that $(k_+(K), k_-(K)) = (0, n)$. On the other hand, as far as the author knows, whether there exist topologically slice knots which satisfy $k_+ > 0$ and $k_- > 0$ remained so far unsolved. In this paper, we give an affirmative answer to the question.

Theorem 2. *For any $m, n \in \mathbb{Z}_{>0}$, there exist infinitely many topologically slice knots with $k_+ \geq m$ and $k_- \geq n$.*

Acknowledgements. The author was supported by JSPS KAKENHI Grant Number 15J10597. The author would like to thank his supervisor, Tamás Kálmán for his encouragement and useful comments. The author also would like to thank Wenzhao Chen, Marco Golla, Matthew Hedden and Jennifer Hom for their stimulating discussions.

2. PRELIMINARIES

In this section, we recall some knot concordance invariants derived from Heegaard Floer homology theory, and show that they give obstructions to sliceness of knots in definite 4-manifolds.

2.1. Correction terms and d_1 -invariant. Ozsváth and Szabó [16] introduced a \mathbb{Q} -valued invariant d (called the *correction term*) for rational homology 3-spheres endowed with a Spin^c structure. In particular, since any integer homology 3-sphere Y has a unique Spin^c structure, we may denote the correction term simply by $d(Y)$ in this case. Furthermore, we note that for any integer homology 3-sphere Y , $d(Y)$ is an even integer.

Let $S_1^3(K)$ denote the 1-surgery along a knot K in S^3 . Then $S_1^3(K)$ is an integer homology 3-sphere, and hence we can define the d_1 -invariant of K as $d_1(K) := d(S_1^3(K))$. It is known that $d_1(K)$ is a knot concordance invariant of K . For details, see [20]. Here we show that the d_1 -invariant gives an obstruction to sliceness in negative definite 4-manifolds.

Lemma 1. *If a knot K is smoothly slice in some negative definite 4-manifold, then we have $d_1(K) = 0$.*

Proof. It is proved in [20] that $d_1(K) \leq 0$ for any knot K . Hence we only need to show that $d_1(K) \geq 0$.

Suppose that K is slice in a negative definite 4-manifold V . Then there exists a properly embedded null-homologous disk D in V with boundary K . By attaching a $(+1)$ -framed 2-handle h^2 along K , and gluing D with the core of h^2 , we obtain an embedded 2-sphere S in $W := V \cup h^2$ with self-intersection $+1$. This implies that there exists a 4-manifold W' with boundary $S_1^3(K)$ such that $W = W' \# \mathbb{C}P^2$. Note that $\partial W' = \partial W = S_1^3(K)$. Since the number of positive eigenvalues of the intersection form of W is one, the intersection form of W' must be negative definite. Now we use the following theorem.

Theorem 3 (Ozsváth-Szabó, [16, Corollary 9.8]). *If Y is an integer homology 3-sphere with $d(Y) < 0$, then there is no negative definite 4-manifold X with $\partial X = Y$.*

By Theorem 3 and the existence of W' , we have $d_1(K) = d(S_1^3(K)) \geq 0$. \square

2.2. τ -invariant, V_k -invariant and ν^+ -invariant. The τ -invariant τ is a famous knot concordance invariant defined by Ozsváth-Szabó [19] and Rasmussen [21]. It is known that τ is a group homomorphism from \mathcal{C} to \mathbb{Z} , while d_1 is not a homomorphism.

The V_k -invariant is a family of $\mathbb{Z}_{\geq 0}$ -valued knot concordance invariants $\{V_k(K)\}_{k \geq 0}$ defined by Ni and Wu [14]. In particular, $\nu^+(K) := \min\{k \geq 0 \mid V_k(K) = 0\}$ is known as the ν^+ -invariant [11]. It is proved in [11] that for any knot K , the inequality $\tau(K) \leq \nu^+(K)$ holds.

In [14], Yi and Ni prove that the set $\{V_k(K)\}_{k \geq 0}$ determines all correction terms of p/q -surgeries along K for any coprime $p, q > 0$. Let $S_{p/q}^3(K)$ denote the p/q -surgery along K . Note that there is a canonical identification between the set of Spin^c structures over $S_{p/q}^3(K)$ and $\{i \mid 0 \leq i \leq p-1\}$. (This identification can be made explicit by the procedure in [18, Section 4, Section 7].)

Proposition 2 (Ni-Wu, [14, Proposition 1.6]). *Suppose $p, q > 0$, and fix $0 \leq i \leq p-1$. Then*

$$d(S_{p/q}^3(K), i) = d(S_{p/q}^3(O), i) - 2 \max \left\{ V_{\lfloor \frac{i}{q} \rfloor}(K), V_{\lfloor \frac{p+q-1-i}{q} \rfloor}(K) \right\},$$

where O denotes the unknot and $\lfloor \cdot \rfloor$ is the floor function.

As a corollary, the following lemma holds. (Note that $\{V_k(K)\}_{k \geq 0}$ satisfy the inequalities $V_k(K) - 1 \leq V_{k+1}(K) \leq V_k(K)$ for each $k \geq 0$.)

Lemma 2. *For any knot K , we have $d_1(K) = -2V_0(K)$.*

Here we show that the τ -invariant also gives an obstruction to sliceness in negative definite 4-manifolds.

Lemma 3. *If a knot K is slice in some negative definite 4-manifold, then we have $\tau(K) \leq 0$.*

Proof. Suppose that K is slice in some negative definite 4-manifold. Then by Lemma 1, we have $d_1(K) = 0$. By Lemma 2, this implies that $V_0(K) = 0$ and $\nu^+(K) = 0$. Hence we have $\tau(K) \leq \nu^+(K) = 0$. \square

By combining Lemma 2 and Lemma 3, we obtain the following obstruction to sliceness in definite 4-manifolds.

Proposition 3. *Let K be a knot. If $d_1(K) \neq 0$ and $\tau(K) < 0$, then K is not slice in any definite 4-manifold.*

Proof. It immediately follows from Lemma 1 that K is not slice in any negative definite 4-manifold. Suppose that K is slice in a positive definite 4-manifold V . Then by reversing the orientation of V , we obtain a slice disk in $-V$ with boundary K^* . Since $-V$ is negative definite and τ is a group homomorphism from \mathcal{C} to \mathbb{Z} , Lemma 3 implies

$$\tau(K) = -\tau(K^*) \geq 0.$$

This contradicts the assumption $\tau(K) < 0$. \square

2.3. Some formulas for V_k -invariants. In this subsection, we recall Wu's cabling formula and Bodnár-Celoria-Golla's connected sum formula for V_k -invariants. Since the $(p, 1)$ -cable and the connected sum of topologically slice knots are also topologically slice, we can estimate the V_k -invariants of various topologically slice knots by using these formulas.

We first recall Wu's cabling formula for V_k . For coprime integers $p, q > 0$, let $T_{p,q}$ denote the (p, q) -torus knot and $K_{p,q}$ the (p, q) -cable of a knot K . We define a map

$$\phi_{p,q} : \left\{ i \mid 0 \leq i \leq \frac{pq}{2} \right\} \rightarrow \left\{ i \mid 0 \leq i \leq q-1 \right\}$$

by

$$\phi_{p,q}(i) \equiv i - \frac{(p-1)(q-1)}{2} \pmod{q}.$$

Proposition 4 (Wu, [22, Lemma 5.1]). *Given $p, q > 0$ and $0 \leq i \leq \frac{pq}{2}$, we have*

$$V_i(K_{p,q}) = V_i(T_{p,q}) + \max \left\{ V_{\lfloor \frac{\phi_{p,q}(i)}{p} \rfloor}(K), V_{\lfloor \frac{p+q-1-\phi_{p,q}(i)}{p} \rfloor}(K) \right\}.$$

(Here we corrected a small mistake in Wu's paper; there should be no factor of 2 in front of the maximum. His proof clearly establishes the formula above.) If we consider the case where $q = 1$, then we have $\phi_{p,1}(i) = 0$ for any $0 \leq i \leq \frac{p}{2}$. Hence Proposition 4 gives the following lemma.

Lemma 4. *Given $p > 0$ and $0 \leq i \leq \frac{p}{2}$, we have*

$$V_i(K_{p,1}) = V_0(K).$$

Next we recall Bodnár-Celoria-Golla's connected sum formula for V_k .

Proposition 5 (Bodnár-Celoria-Golla, [1, Proposition 6.1]). *For any two knots K and J and any $m, n \in \mathbb{Z}_{\geq 0}$, we have*

$$V_{m+n}(K \# J) \leq V_m(K) + V_n(J).$$

Here I would like to point out that in the proof of this proposition for the cases where $m = 0$ or $n = 0$, the authors of [1] apply Ni-Wu's formula (i.e, Proposition 2 in the present paper) to $S_0^3(K)$ and $S_0^3(J)$ without any comment, although Ni-Wu's formula is only proved for $S_{p/q}^3(K)$ with coprime $p, q > 0$. For completeness, we give a proof of Proposition 5 for the cases where $m = 0$ or $n = 0$.

Proof of Proposition 5. We first consider the case where $m = 0$ and $n > 0$. Let W be a 4-manifold represented by the relative diagram in Figure 1, which is an oriented cobordism from $S_2^3(K) \# S_{2n}^3(J)$ to $S_{2n+2}^3(K \# J)$. (For details of relative diagrams, see [7].) Furthermore, we define X to be a 4-manifold represented by the diagram in Figure 1 with all brackets deleted, and \tilde{X} to be the closure of $X \setminus W$. It follows from elementary algebraic topology that $H_2(W; \mathbb{Z}) \cong \mathbb{Z}$ and $\sigma(W) = \sigma(X) - \sigma(\tilde{X}) = -1$, hence W is a negative definite. We use the following lemma, which immediately follows from [16, Theorem 9.6].

Lemma 5 (Ozsváth-Szabó). *Let Y_1 and Y_2 be rational homology 3-spheres and W a negative definite cobordism from Y_1 to Y_2 . Then for any Spin^c structure \mathfrak{s} over W , we have the inequality*

$$c_1(\mathfrak{s})^2 + \beta_2(W) \leq 4d(Y_2, \mathfrak{s}|_{Y_2}) - 4d(Y_1, \mathfrak{s}|_{Y_1}).$$

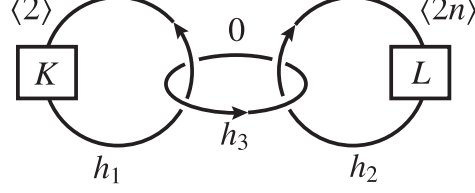


FIGURE 1.

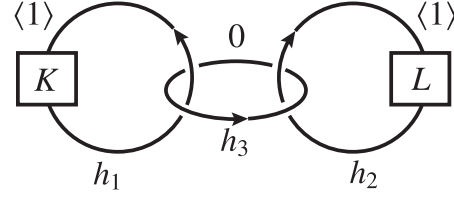


FIGURE 2.

To apply Lemma 5 to W , we take a Spin^c structure over W as follows. By choosing the order and the orientation of generators of $H_2(X; \mathbb{Z})$ as shown in Figure 1, we have a representation matrix

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2n & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

for the intersection form $Q_X : H_2(X; \mathbb{Z}) \times H_2(X; \mathbb{Z}) \rightarrow \mathbb{Z}$. Let h_i^* denote the cocore of h_i ($i = 1, 2, 3$). We take a Spin^c structure \mathfrak{s} over X such that $PD(c_1(\mathfrak{s})) = (-2, 0, 0) \in H_2(X, \partial X; \mathbb{Z})$ with respect to the basis $\{h_1^*, h_2^*, h_3^*\}$. For the restriction of \mathfrak{s} to W , we have the inequality

$$(1) \quad c_1(\mathfrak{s}|_W)^2 + 1 \leq 4d(S_2^3(K \# J), \mathfrak{s}|_{S_{2n+2}^3(K \# J)}) - 4d(S_2^3(K), \mathfrak{s}|_{S_2^3(K)}) - 4d(S_{2n}^3(J), \mathfrak{s}|_{S_{2n}^3(J)}).$$

Here we compute $c_1(\mathfrak{s}|_W)^2$. Let $Q_{\tilde{X}}$ and Q_W denote the intersection form of \tilde{X} and W respectively. Since the inclusion maps induce an isomorphism $H_2(X; \mathbb{Q}) \cong H_2(\tilde{X}; \mathbb{Q}) \oplus H_2(W; \mathbb{Q})$ and Q_X decomposes as $Q_{\tilde{X}} \oplus Q_W$ over \mathbb{Q} , we have

$$c_1(\mathfrak{s})^2 = c_1(\mathfrak{s}|_{\tilde{X}})^2 + c_1(\mathfrak{s}|_W)^2.$$

Furthermore, it is not hard to check that $c_1(\mathfrak{s})^2 = 2/(n+1)$ and $c_1(\mathfrak{s}|_{\tilde{X}})^2 = 2$. Hence we have $c_1(\mathfrak{s}|_W)^2 = -2n/(n+1)$. Next we consider which integers correspond to $\mathfrak{s}|_{S_2^3(K)}$, $\mathfrak{s}|_{S_{2n}^3(J)}$ and $\mathfrak{s}|_{S_{2n+2}^3(K \# J)}$ respectively. For a knot L and $k \in \mathbb{Z}$, let $X_k(L)$ denote a 4-manifold obtained by attaching a single 2-handle h_L^2 to B^4 along a knot $K \# L \subset \partial B^4$ with framing k . In addition, let F_L denote a closed surface obtained

by gluing the core of h_L^2 to a Seifert surface for L . Then we note that \tilde{X} and X are diffeomorphic to $X_2(K) \natural X_{2n}(J)$ and $X_{2n+2}(K \# J) \# S^2 \times S^2$ respectively (where \natural denotes boundary connected sum). Furthermore, we can see that $[F_K] = (1, 0, 0)$, $[F_J] = (0, 1, 0)$ and $[F_{K \# J}] = (1, -1, 2n)$ with respect to the basis $\{[h_1], [h_2], [h_3]\}$. Hence we have

$$\langle c_1(\mathfrak{s}), [F_K] \rangle = -2, \langle c_1(\mathfrak{s}), [F_J] \rangle = 0 \text{ and } \langle c_1(\mathfrak{s}), [F_{K \# J}] \rangle = -2.$$

This implies that $\mathfrak{s}|_{S_2^3(K)}$, $\mathfrak{s}|_{S_{2n}^3(J)}$ and $\mathfrak{s}|_{S_{2n+2}^3(K \# J)}$ are identified with 0, n and n respectively. Now we can reduce the inequality (1) to

$$\begin{aligned} -\frac{2n}{n+1} + 1 &\leq 4d(S_{2n+2}^3(O), n) - 4d(S_2^3(O), 0) - 4d(S_{2n}^3(O), n) \\ &\quad - 8V_n(K \# J) + 8V_0(K) + 8V_n(J). \end{aligned}$$

Since $4d(S_{2n+2}^3(O), n) = 1 - 2n/(n+1)$, $4d(S_2^3(O), 0) = 1$ and $4d(S_{2n}^3(O), n) = -1$, we have the desired inequality $V_n(K \# J) \leq V_0(K) + V_n(J)$.

For the case where $m = n = 0$, it suffices to apply the above argument after replacing Figure 1 with Figure 2, taking a Spin^c structure \mathfrak{s} over X such that $PD(c_1(\mathfrak{s})) = (1, 1, 2)$. \square

In this paper, we only need Proposition 5 in the case where $m = n = 0$, which is stated as follows.

Proposition 6. *For any two knots K and J , we have*

$$V_0(K \# J) \leq V_0(K) + V_0(J).$$

We can use Proposition 6 to give a lower bound for V_0 of the connected sum of two knots as well. In particular, we have the following lemma.

Lemma 6. *For any two knots K and J , we have*

$$V_0(K \# J^*) \geq V_0(K) - V_0(J).$$

Proof. For the inequality in Proposition 6, by replacing K with $K \# J^*$, we have

$$V_0(K \# J^* \# J) \leq V_0(K \# J^*) + V_0(J).$$

Since $K \# J^* \# J$ is concordant to K , we have $V_0(K \# J^* \# J) = V_0(K)$. This completes the proof. \square

3. PROOF OF THE MAIN THEOREMS

In this section, we prove Proposition 1, Theorem 1 and Theorem 2. We first prove Proposition 1 and Theorem 1.

Proof of Proposition 1. Suppose that two given knots K and J satisfy $V_0(K) > V_0(J)$ and $\tau(K), \tau(J) > 0$. Fix a positive integer n with $\tau(K) < n \cdot \tau(J)$. Then Lemma 4 and Lemma 6 imply that

$$V_0(K \# (J_{n,1})^*) \geq V_0(K) - V_0(J_{n,1}) = V_0(K) - V_0(J) > 0.$$

Hence by Lemma 2, we have $d_1(K \# (J_{n,1})^*) = -2V_0(K \# (J_{n,1})^*) \neq 0$. Furthermore, by [9, Theorem 1.2], we have

$$\tau(K \# (J_{n,1})^*) = \tau(K) - \tau(J_{n,1}) \leq \tau(K) - n \cdot \tau(J) < 0.$$

Hence it follows from Proposition 3 that $K \# (J_{n,1})^*$ is not slice in any definite 4-manifold. \square

Proof of Theorem 1. For a knot K , let $\text{Wh}(K)$ denote the positively-clasped un-twisted Whitehead double of K . Then we set

$$K_n := (\#_{i=1}^3 \text{Wh}(T_{2,3})) \# ((\text{Wh}(T_{2,3}))_{n+3,1})^*$$

for any positive integer n . Since the Alexander polynomial of K_n equals 1, K_n is topologically slice for any n . [4, 5]

We first prove that K_n is not slice in any definite 4-manifold. By Lemma 4 and Lemma 6, we have

$$V_0(K_n) \geq V_0(\#_{i=1}^3 \text{Wh}(T_{2,3})) - V_0(\text{Wh}(T_{2,3}))$$

for any n . As mentioned in [12, Section 3], it is proved in [10, Theorem B.1] that $\#_{i=1}^k \text{Wh}(T_{2,3})$ is ν^+ -equivalent to $T_{2,2k+1}$ for any $k > 0$. (Here, knots K and J being ν^+ -equivalent means that the equalities $\nu^+(K \# J^*) = \nu^+(J \# K^*) = 0$ hold.) Hence it follows from [12, Lemma 3.1], the definition of V_k and [17, Corollary 1.5] that $V_0(\#_{i=1}^k \text{Wh}(T_{2,3})) = V_0(T_{2,2k+1}) = \lceil \frac{k}{2} \rceil$. This implies that

$$V_0(K_n) \geq \left\lceil \frac{3}{2} \right\rceil - \left\lceil \frac{1}{2} \right\rceil = 1 > 0.$$

In particular, we have $d_1(K_n) \neq 0$. Moreover, it follows from [8, Theorem 1.5] that $\tau(\text{Wh}(T_{2,3})) = 1$, and hence [9, Theorem 1.2] implies that

$$\tau(K_n) = 3 - (n + 3) = -n < 0.$$

Therefore, it follows from Proposition 3 that for any $n > 0$, the knot K_n is not slice in any definite 4-manifold.

Next we prove that the knots $\{K_n\}_{n \in \mathbb{Z}_{>0}}$ are linearly independent. Suppose that a linear combination $m_1[K_{n_1}] + \dots + m_k[K_{n_k}]$ equals zero in \mathcal{C} (where we may assume that $0 < n_1 < n_2 < \dots < n_k$). Then we have the equality

$$(2) \quad 3(\sum_{i=1}^k m_i)[\text{Wh}(T_{2,3})] = m_1[(\text{Wh}(T_{2,3}))_{n_1+3,1}] + \dots + m_k[(\text{Wh}(T_{2,3}))_{n_k+3,1}].$$

In the proof of [12, Theorem A], the authors define a homomorphism $\phi : \mathcal{C} \rightarrow \mathbb{Z}^\infty$ and show that $\phi([(\text{Wh}(T_{2,3}))_{n+3,1}]) = (*, \dots, *, 1, 0, 0, \dots)$ where 1 is the $(n+2)^{\text{nd}}$ coordinate. Hence we can see that the $(n_k + 2)^{\text{nd}}$ coordinate of $\phi(\text{RHS of (2)})$ is m_k . On the other hand, we can verify that $\phi([\text{Wh}(T_{2,3})]) = \phi([T_{2,3}]) = (0, 0, \dots)$, and hence m_k must be 0. Inductively, we have $m_1 = \dots = m_k = 0$. This completes the proof. \square

Remark. If we do not require knots in Theorem 1 to be topologically slice, then the existence of such a family can be established using the following proposition, which immediately follows from [2, Proposition 1.2].

Proposition 7. *If the Levine-Tristram signature of a knot K has both positive and negative values, then K is not smoothly slice in any definite 4-manifold.*

Indeed, we can take $J_k := \{T_{2,2k+9} \# (\#_{i=1}^{k+5} T_{2,3})^*\}_{k \in \mathbb{Z}_{>0}}$ as the concrete sequence. (we can verify that $\sigma_{J_k}(e^{i\theta}) = -2$ for $\theta \in (\frac{\pi}{2k+9}, \frac{3\pi}{2k+9})$ and $\sigma_{J_k}(-1) = 2$. Furthermore, since all torus knots are linearly independent in \mathcal{C} [13], the knots J_k are also linearly independent.)

Finally we prove Theorem 2. To do so, we use the following observation relating kinkiness to ν^+ and τ .

Lemma 7. *For any knot K , we have the inequalities*

$$\nu^+(K) \leq k^+(K)$$

and

$$-k^-(K) \leq \tau(K) \leq k^+(K).$$

Proof. If a knot K_1 is deformed into K_2 by a crossing change from a positive crossing (Figure 3) to a negative crossing (Figure 4) (resp. from a negative crossing to a positive crossing), then we say that K_1 is deformed into K_2 by a *positive (resp. negative) crossing change*. It is proved in [1, Theorem 1.3] and [19, Corollary 1.5] that if a knot K_+ is deformed into K_- by a positive crossing change, then we have

$$\nu^+(K_-) \leq \nu^+(K_+) \leq \nu^+(K_-) + 1$$

and

$$\tau(K_-) \leq \tau(K_+) \leq \tau(K_-) + 1.$$

Furthermore, it follows from [15, Proposition 2.1] that for any knot K , there exists a knot J so that J is concordant to K and J can be deformed into a slice knot L by just $k^+(K)$ positive crossing changes and finitely many negative crossing changes. These imply that

$$\nu^+(K) = \nu^+(J) \leq \nu^+(L) + k^+(K) = k^+(K)$$

and

$$\tau(K) = \tau(J) \leq \tau(L) + k^+(K) = k^+(K).$$

By applying the same argument to K^* , we have

$$-\tau(K) = \tau(K^*) \leq k^+(K^*) = k^-(K).$$

□

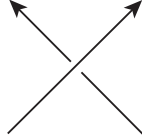


FIGURE 3.

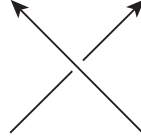


FIGURE 4.

Proof of Theorem 2. For positive integers k and l , we define $K_{k,l}$ by

$$K_{k,l} := (\#_{i=1}^{2k+1} \text{Wh}(T_{2,3})) \# ((\text{Wh}(T_{2,3}))_{l+2k+1,1})^*.$$

Obviously, $K_{k,l}$ is topologically slice for any $k, l > 0$. We prove that for any $m, n \in \mathbb{Z}_{>0}$, $\{K_{m,l}\}_{l \geq n}$ are mutually distinct and each of them satisfies $k^+(K_{m,l}) \geq m$ and $k^-(K_{m,l}) \geq n$.

By applying the argument in the proof of Theorem 1, we have

$$V_0(K_{k,l}) \geq V_0(\#_{i=1}^{2k+1} \text{Wh}(T_{2,3})) - V_0((\text{Wh}(T_{2,3}))_{l+2k+1,1}) = k$$

and

$$\tau(K_{k,l}) = 2k + 1 - (l + 2k + 1) = -l.$$

In particular, $K_{k,l}$ is not equal to $K_{k,l'}$ if $l \neq l'$. Furthermore, since $\nu^+(K) = \min\{i \in \mathbb{Z}_{\geq 0} | V_i(K) = 0\}$ and $V_{i+1}(K) \geq V_i(K) - 1$, we have $\nu^+(K_{k,l}) \geq k$. Hence Lemma 7 proves that $k^+(K_{k,l}) \geq k$ and $k^-(K_{k,l}) \geq l$. This completes the proof. \square

REFERENCES

- [1] J. Bodnár, D. Celoria and M. Golla, *A note on cobordisms of algebraic knots*. arXiv:1509.08821 (2015).
- [2] T. D. Cochran, S. Harvey and P. Horn, *Filtering smooth concordance classes of topologically slice knots*. *Geom. Topol.* 17 (2013), no. 4, 2103–2162.
- [3] H. Endo, *Linear independence of topologically slice knots in the smooth cobordism group*. *Topology Appl.* 63 (1995), no. 3, 257–262.
- [4] M. Freedman, *The topology of four-dimensional manifolds*. *J. Differential Geom.* 17 (1982), no. 3, 357–453.
- [5] M. Freedman and F. Quinn, *Topology of 4-manifolds*. Princeton Mathematical Series, 39. Princeton University Press, Princeton, NJ, 1990.
- [6] R. E. Gompf, *Smooth concordance of topologically slice knots*. *Topology* 25 (1986), no. 3, 353–373.
- [7] R. E. Gompf and A. I. Stipsicz, *4-manifolds and Kirby calculus*. Graduate Studies in Mathematics, 20. American Mathematical Society, Providence, RI, 1999.
- [8] M. Hedden, *Knot Floer homology of Whitehead doubles*. *Geom. Topol.* 11 (2007), 2277–2338.
- [9] M. Hedden, *On knot Floer homology and cabling. II*. *Int. Math. Res. Not. IMRN* 2009, no. 12, 2248–2274.
- [10] M. Hedden, S. G. Kim and C. Livingston, *Topologically slice knots of smooth concordance order two*. *J. Differential Geom.* 102 (2016), no. 3, 353–393.
- [11] J. Hom and Z. Wu, *Four-ball genus bounds and a refinement of the Ozsváth-Szabó tau-invariant*. arXiv:1401.1565 (2014).
- [12] M. H. Kim and K. Park, *An infinite-rank summand of knots with trivial Alexander polynomial*. arXiv:1604.04037 (2016).
- [13] R. A. Litherland, *Signatures of iterated torus knots*. *Topology of low-dimensional manifolds* (Proc. Second Sussex Conf., Chelwood Gate, 1977), pp. 71–84, *Lecture Notes in Math.*, 722, Springer, Berlin, 1979.
- [14] Y. Ni and Z. Wu, *Cosmetic surgeries on knots in S^3* . *J. Reine Angew. Math.* 706 (2015), 1–17.
- [15] B. Owens and S. Strle, *Immersed disks, slicing numbers and concordance unknotting numbers*. arXiv:1311.6702 (2013).
- [16] P. Ozsváth and Z. Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*. *Adv. Math.* 173 (2003), 179–261.
- [17] P. Ozsváth and Z. Szabó, *Heegaard Floer homology and alternating knots*. *Geom. Topol.* 7 (2003), 225–254.
- [18] P. Ozsváth and Z. Szabó, *Knot Floer homology and rational surgeries*. *Algebr. Geom. Topol.* 11 (2011), no. 1, 1–68.
- [19] P. Ozsváth and Z. Szabó, *Knot Floer homology and the four-ball genus*. *Geom. Topol.* 7 (2003), 615–639.
- [20] T. D. Peters, *A concordance invariant from the Floer homology of ± 1 surgeries*. arXiv:1003.3038 (2010).
- [21] J. Rasmussen, *Floer homology and knot complements*. arXiv:0306.378 (2003).
- [22] Z. Wu, *A cabling formula for ν^+ invariant*. arXiv:1501.04749 (2015).